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THE PERIODIC THERMALLY DEVELOPED REGIME IN DUCTS WITH STREAMWISE PERIODIC WALL TEMPERATURE OR HEAT FLUX

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Abstract—When a tube or duct flow is subjected to a streamwise periodic variation of wall temperature or wall heat flux, a periodic thermally developed regime is established at sufficient distances from the inlet. In the developed regime, similar heat-transfer characteristics occur in successive streamwise modules whose length is equal to the period of the imposed variation. A methodology was developed for obtaining universal solutions for the periodic fully developed regime. The end result of the development enables the wall heat flux distribution corresponding to any given wall temperature variation to be determined via the summing of a simple series. A similar series enables the wall temperature to be determined when the heat flux distribution is given. The series contains influence coefficients which represent the solution for an array of pulses. A procedure for determining these influence coefficients is presented which circumvents the need to solve the differential energy equation. The use of the methodology was illustrated by application to sine wave wall temperature and heat flux distributions.

NOMENCLATURE

c_p, specific heat at constant pressure;
f, heat flux for entrance region problem, equation (18);
g, wall temperature for entrance region problem, equation (23);
k, thermal conductivity;
L, period length of the imposed variation, module length;
$\mathscr{L}$, dimensionless length of module, $(L/r_0)/Pe$;
l, dimensionless pulse length, $\mathscr{L}/N$;
$\dot{m}$, mass rate of flow;
N, number of subdivisions in a module;
Pe, Peclet number, $\bar{u}(2r_0)/\alpha$;
Q', local wall heat-transfer rate per unit length;
q, local wall heat flux;
r, radial coordinate;
r_0, tube radius;
T, temperature relative to any datum such that $T_w \neq 0$;
T_b, bulk temperature;
T_b^+, bulk temperature at x^+;
ΔT_b, bulk temperature rise per module;
T_w, wall temperature;
u, axial velocity;
$\bar{u}$, mean velocity;
X, dimensionless axial coordinate, $(x/r_0)/Pe$;
x, axial coordinate;
x^+, beginning of module of prescribed T_w or q variation;
x^*, beginning of module of basic pulse array.

Greek symbols
α, thermal diffusivity;
Λ, influence coefficients, equation (11);
ρ, density;
Ω, influence coefficients, equation (7).

INTRODUCTION

IN CONVENTIONAL duct flows, the thermally developed regime is characterized by a heat-transfer coefficient that is independent of the streamwise coordinate. In the literature, a number of thermal boundary conditions have been identified which give rise to the developed regime. The best known among these are uniform wall temperature and uniform heat addition (or removal) per unit length. Thermal development is also attained when there is convective heat exchange between the external surface of the duct and a fluid environment having a uniform heat-transfer coefficient and uniform temperature. A fourth condition which yields a developed regime is an exponential variation ($\sim e^{\beta x}$) of the heat transfer per unit length.† From the foregoing, it is seen that the boundary conditions which define the conventional thermally developed regime are either constant in the streamwise direction or vary monotonically as an exponential.

One of the features of the thermally developed regime is that the heat-transfer coefficients can be obtained by solving equations which are substantially simplified versions of the conservation laws. Further-

†In [1], it is shown that uniform wall temperature, uniform heat addition, and external convection all correspond to heat-transfer distributions that are special cases of the exponential variation.

more, the thermally developed results can be determined without having to deal with the relatively difficult entrance region problem.

Recently [2], the concepts of fully developed flow and heat transfer were generalized to accommodate ducts whose cross-sectional area varies periodically in the streamwise direction. If L denotes the streamwise length of each cycle of the area variation, then successive streamwise modules of length L possess common fluid flow and heat-transfer characteristics. In particular, the distribution of the heat-transfer coefficient on the duct surfaces contained in any one module is repeated in all other modules in the fully developed regime. This characteristic enables the analysis of the developed regime to be confined to a single typical module, without involvement with the entrance region problem.

In the present paper, a further generalization of the thermally developed regime is formulated and illustrated via applications. Attention is focussed here on flows in which there is a prescribed periodic variation of either the wall temperature or the wall heat transfer along the length of the duct. In particular, we consider straight ducts of constant cross-section in which the flow becomes hydrodynamically developed at sufficient distances from the inlet. Such flows will experience a periodic thermally developed regime, the nature of which will be described shortly. This periodic regime, which exists in response to the thermal boundary conditions, is in contrast to that of [2] which is caused by flow periodicity.

The thermally developed regime to be described here pertains to tubes and ducts for which the wall temperature T_w is circumferentially uniform but varies in the streamwise direction. Consider first the case in which T_w varies periodically with the streamwise coordinate x as illustrated schematically in Fig. 1,

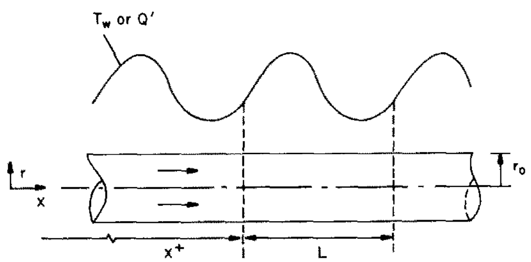

FIG. 1. A tube subjected to streamwise periodic variations of wall temperature or wall heat flux.

where L denotes the period length. At sufficiently large downstream distances, the thermal history experienced by the fluid in any period length, for example, between $x = x^+$ and $x = (x^+ + L)$ has to be identical to that experienced in all similar modules. From this, it follows that the temperature distributions at $x = x^+$, $x = (x^+ + L)$, $x = (x^+ + 2L), \ldots$, are the same. Since these temperature distributions are identical, the net rate of heat transfer per module is zero, that is, the inflows and outflows along the length of the module are in balance. These characteristics will be employed to formulate the analysis of the thermally developed regime for the prescribed temperature boundary condition.

Next, suppose that the rate of heat transfer per unit length Q' is given a prescribed variation with x which is once again illustrated by Fig. 1. Between two stations such as x^+ and $(x^+ + L)$, heat is added to (or removed from) the fluid at the rate

$$Q = \int_{x^+}^{x^+ + L} Q' \, dx. \tag{1}$$

Corresponding to Q, there is a bulk temperature rise (or fall) ΔT_b given by

$$\Delta T_b = Q/\dot{m}c_p. \tag{2}$$

In the thermally developed regime, the temperature distribution at $(x^+ + L)$ is equal to that at x^+ plus the additive constant ΔT_b. This relation, which holds at any pair of stations in the thermally developed regime that are separated by the period length L, forms the basis of the mathematical analysis.

As will soon be demonstrated, the analysis of the periodic thermally developed regime can be confined to a typical module of period length L and need not be concerned with the entrance region. The governing differential equation (i.e. the energy equation) and the boundary conditions for the module will be discussed in the next section. In general, numerical solutions are required. Whereas there appear to be no insuperable difficulties associated with the execution of the numerical solutions, there is a major drawback in a direct numerical attack on the problem. In particular, a specific numerical solution would have to be carried out for each prescribed periodic distribution of T_w or Q'. It appears preferable to formulate a methodology capable of dealing with any arbitrary periodic distribution without having to solve the governing differential equation.

The development and application of such a methodology is the main focus of this paper. The essence of the method is to envision the prescribed periodic distribution as being made up of an assemblage of pulses which approximate the distribution curve. From the assemblage, a fundamental array of periodically positioned pulses is identified. The solution for the fundamental array provides a table of influence coefficients from which the results corresponding to any prescribed periodic distribution can be synthesized.

For concreteness, the methodology is developed in terms of laminar flow in a circular tube, and corresponding tabulations of influence coefficients are presented. The use of the methodology and the tabulated information is illustrated by application to the cases where the wall temperature and wall heat transfer vary sinusoidally along the tube. It is worthy of note, however, that the approach can also be employed for turbulent flow as well as for other types of ducts.

THE SOLUTION METHOD

The most direct approach to the analysis of the periodic thermally developed regime is to solve the energy equation for a typical module. For constant fluid properties and for negligible axial conduction and viscous dissipation, the energy equation for laminar hydrodynamically developed pipe flow is

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3}$$

where the coordinates are illustrated in Fig. 1. To complete the mathematical description of the problem, equation (3) has to be supplemented by boundary conditions at the tube wall and periodicity conditions at the upstream and downstream ends of the module. For concreteness, attention will be focused on the module lying between $x = x^+$ and $x = (x^+ + L)$ in Fig. 1.

For prescribed periodically varying wall temperature, the boundary and periodicity conditions are, respectively

$$T_w = T_w(x), \quad x^+ \leq x \leq (x^+ + L) \tag{4a}$$

$$T[(x^+ + L), r] = T(x^+, r), \quad 0 \leq r \leq r_0. \tag{4b}$$

Here, and throughout the rest of the paper, the temperature T is taken relative to any datum such that $T_w(x) \neq 0$. When the heat transfer is prescribed and the heating is uniform around the circumference of the tube, then the local flux q is equal to $Q'/2\pi r_0$. For periodic q (or Q')

$$q = q(x) = k(\partial T/\partial r)_w, \; x^+ \leq x \leq (x^+ + L) \tag{5a}$$

$$T[(x^+ + L), r] = T(x^+, r) + \Delta T_b, 0 \leq r \leq r_0 \tag{5b}$$

where ΔT_b is the bulk rise per module as given by equation (2). It is the periodicity conditions (4b) and (5b) which enable the fully developed regime to be solved without having to deal with the entrance region.

In general, equation (3) would have to be solved numerically and, except for isolated cases, the numerical problem is two dimensional and elliptic. Although there is ample experience indicating that such problems can be solved, the fact remains that a new solution would have to be carried out for each case.

As an alternative, a methodology will now be developed which provides universal solutions applicable to any prescribed periodic wall temperature or periodic heat flux.

Universal solutions—prescribed temperature

The linearity of the energy equation (3) and its boundary and periodicity conditions permits superposition. To this end, as illustrated in the upper diagram of Fig. 2, the periodic wall temperature distribution can be regarded as being made up of an assemblage of temperature pulses. In the period length L, there are N such pulses, each of width L/N.

Figure 2 illustrates how the total assemblage of pulses can be broken down into N arrays. Each array

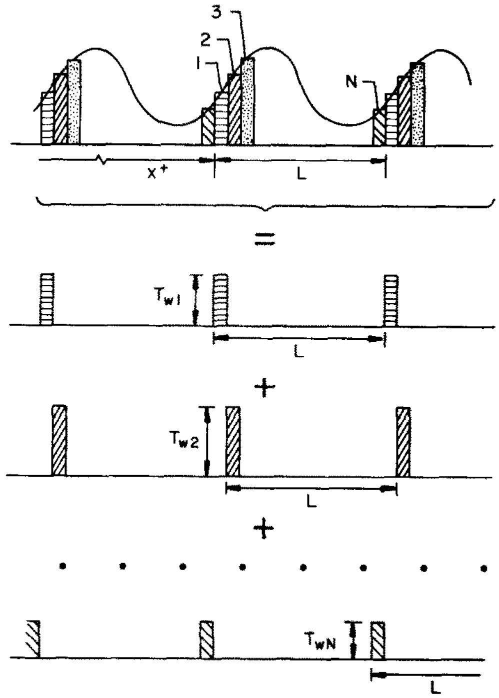

FIG. 2. Representation of the periodic distribution by an assemblage of pulses and by N arrays of pulses.

consists of a succession of pulses, each of height T_{wn} ($n = 1, 2, \ldots, N$) and width L/N, which are spaced apart by the period length L. Clearly, the superposition of the N arrays results in the total assemblage of pulses which represents the given periodic distribution. Furthermore, the solutions for the N arrays, when superposed, give the solution to the given problem to an accuracy that is governed by the size of the pulse width.

Although there are N solutions to be superposed, it is not necessary to generate N independent solutions. This is because each of the N arrays represents the same problem, the only distinctions being a linear displacement along the x-axis and a different pulse height. Therefore, only a single solution need be obtained—that corresponding to a typical array of pulses (i.e. any one of the arrays pictured in Fig. 2). By proper manipulation of the solution of this basic problem, the solutions for all N arrays can be obtained.

The basic problem is pictured in Fig. 3, where a dimensionless axial coordinate X and dimensionless lengths $\mathscr{L}$ and l are used. These are defined as

$$X = (x/r_0)/Pe, \quad \mathscr{L} = (L/r_0)/Pe, \quad l = \mathscr{L}/N. \tag{6}$$

Furthermore, if T_w represents the pulse height, then T/T_w may be used as the dimensionless temperature

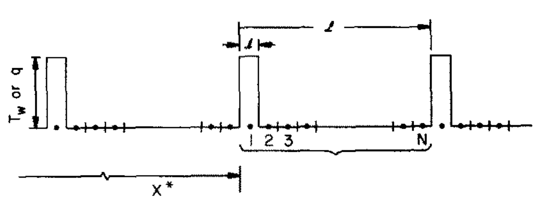

FIG. 3. The basic pulse array.

variable. These quantities, along with a dimensionless radial coordinate r/r_0, may be introduced into the energy equation and the boundary and periodicity conditions for the array of pulses. From this, it is found that the problem is governed by only one parameter, $\mathscr{L}$.

There are various options that might be considered for solving the basic pulse problem, one of which is to obtain direct numerical solutions. There is, however, a simpler alternative, as will be described later in the paper. For the present, it will be assumed that solutions are available for various parametric values of $\mathscr{L}$.

The solution yields the distribution of the heat flux q along a typical module of the pulse array of Fig. 3. The pulse module need not coincide with the module in the given temperature distribution that is pictured at the top of Fig. 2, that is, X^* and X^+ may be different. Indeed, this non-coincidence enables the solution for the basic pulse array to serve for all N pulse arrays shown beneath the equals sign in Fig. 2.

For the basic array of Fig. 3, the solution gives the heat fluxes $q_1, q_2, \ldots, q_N$ at the midpoints $1, 2, \ldots, N$ of the segments which make up the module. These results are in dimensionless form in terms of

$$q_n T_0/kT_w \equiv \Omega_n, \quad 1 \leqslant n \leqslant N \tag{7}$$

and are parameterized by $\mathscr{L}$. The Ω_n will be designated as influence coefficients. For future reference, it should be emphasized that the subscript n indicates the location at which the influence of the pulse is being expressed through Ω_n. The numerical value of n is the serial number of the segment of interest, counting the pulsed segment as $n = 1$.

It will now be demonstrated how the influence coefficients are employed to obtain the heat flux distribution in a typical module of the temperature distribution shown at the top of Fig. 2. In this connection, let the points $1, 2, \ldots, N$ be positioned at the midpoints of the successive segments which make up the module. The heat fluxes at these points will be denoted $q(1), q(2), \ldots, q(j), \ldots, q(N)$.

First, the contribution to the q's due to the array of pulses of height T_{w1} will be determined. For this, the basic array of pulses may be regarded as positioned so that $X^* = X^+$. Correspondingly,

$$q(1) = (kT_{w1}/r_0)\Omega_1,\ q(2) = (kT_{w1}/r_0)\Omega_2, \ldots \tag{8}$$

Next, to obtain the contribution due to the array of pulses of height T_{w2}, the basic array is positioned so that $X^* = (X^+ + l)$. For this situation,

$$\begin{aligned} q(1) &= (kT_{w2}/r_0)\Omega_N, \\ q(2) &= (kT_{w2}/r_0)\Omega_1, \\ q(3) &= (kT_{w2}/r_0)\Omega_2, \ldots \end{aligned} \tag{9}$$

Since the pulse is situated at segment 2, the proper influence coefficient for $q(2)$ *is* Ω_1, and the other influence coefficients are taken relative to this pulse position. For the array of pulses of height T_{w3}, the basic array is positioned with $X^* = X^+ + 2l$. The successive Ω's used for $q(1), q(2), q(3), \ldots$ are $\Omega_{N-1}, \Omega_N, \Omega_1, \ldots$, and so on.

It remains to combine the contributions from the various arrays of pulses. When the combination is performed, there is obtained

$$q(j)r_0/k = \sum_{n=1}^{j} \Omega_{j+1-n} T_{wn} + \sum_{n=j+1}^{N} \Omega_{N+j+1-n} T_{wn} \tag{10}$$

for $j = 1, 2, \ldots, N$. Once the influence coefficients are known, equation (10) can be used to compute the wall heat flux distribution for any given periodic wall temperature distribution $T_w(x)$.

As written, equation (10) is not restricted to laminar tube flows. It is applicable to any flow situation with prescribed periodic wall temperature for which the influence coefficients can be determined (r_0 would be replaced by an appropriate characteristic dimension).

A listing of influence coefficients for laminar tube flow is presented in Table 1. The table contains results for five values of the dimensionless module length $\mathscr{L} = (L/r_0)/Pe$ equal to 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, and 1. For each of these, 20 influence coefficients are listed (i.e. $N = 20$). The motivation for selecting $N = 20$ is that this number of subdivisions, in addition to providing an adequate representation of most practical variations, enables the influence coefficients to be used in subsequent calculations without the aid of a digital computer. The first coefficient in each set is positive whereas the others are negative and decrease in magnitude with increasing n. The method by which these coefficients were determined will be described later. Their numerical application will be illustrated shortly.

As a final remark about equation (10), it may be noted that since $q(j) = 0$ for uniform wall temperature, it follows that the influence coefficients sum to zero. As a consequence, a constant value can be added to all of the T_{wn} without affecting $q(j)$. Therefore, as expected, q is independent of the datum of the temperature.

Universal solutions—prescribed heat flux

The construction of the universal solutions for the case of prescribed periodic wall heat flux is carried out in a manner similar to that which has just been presented for prescribed wall temperature. Figure 2 may be taken as the starting point of the derivation for the prescribed heat flux case, where the distribution curve at the top of the figure and the various pulse arrays now represent heat flux rather than temperature. Furthermore, Fig. 3 now depicts a basic array of heat flux pulses, the solution for which can be utilized to generate the solutions for all N of the pulse arrays pictured in Fig. 2.

For the basic array, the solution gives the local wall-to-bulk temperature difference at the midpoints $1, 2, \ldots, N$ of the segments which make up the typical module. These results are expressed as

$$(T_w - T_b)_n/(qr_0/k) \equiv \Lambda_n, \quad 1 \leqslant n \leqslant N. \tag{11}$$

The Λ_n (the influence coefficients) depend parametrically on $\mathscr{L} = (L/r_0)/Pe$.

Table 1. Influence coefficients† Ω_n

n	$(L/r_0)/Pe =$ 0.0001	0.001	0.01	0.1	1.0
1	0.48532(+2)	0.22216(+2)	0.99999(+1)	0.43214(+1)	0.16611(+1)
2	−0.16147(+2)	−0.74815(+1)	−0.34590(+1)	−0.15958(+1)	−0.75795(0)
3	−0.62273(+1)	−0.28764(+1)	−0.13201(+1)	−0.59974(0)	−0.29958(0)
4	−0.38842(+1)	−0.17886(+1)	−0.81465(0)	−0.36376(0)	−0.18765(0)
5	−0.28657(+1)	−0.13157(+1)	−0.59482(0)	−0.26090(0)	−0.12802(0)
6	−0.23094(+1)	−0.10553(+1)	−0.47374(0)	−0.20411(0)	−0.88580(−1)
7	−0.19528(+1)	−0.89180(0)	−0.39767(0)	−0.16832(0)	−0.61425(−1)
8	−0.17120(+1)	−0.77997(0)	−0.34562(0)	−0.14377(0)	−0.42610(−1)
9	−0.15371(+1)	−0.69877(0)	−0.30782(0)	−0.12587(0)	−0.29559(−1)
10	−0.14044(+1)	−0.63713(0)	−0.27911(0)	−0.11224(0)	−0.20506(−1)
11	−0.13002(+1)	−0.58870(0)	−0.25654(0)	−0.10149(0)	−0.14225(−1)
12	−0.12160(+1)	−0.54961(0)	−0.23832(0)	−0.92780(−1)	−0.98686(−2)
13	−0.11466(+1)	−0.51733(0)	−0.22326(0)	−0.85771(−1)	−0.68461(−2)
14	−0.10882(+1)	−0.49020(0)	−0.21060(0)	−0.79493(−1)	−0.47493(−2)
15	−0.10383(+1)	−0.46703(0)	−0.19979(0)	−0.74288(−1)	−0.32947(−2)
16	−0.99513(0)	−0.44698(0)	−0.19042(0)	−0.69771(−1)	−0.22856(−2)
17	−0.95737(0)	−0.42943(0)	−0.18223(0)	−0.65802(−1)	−0.15856(−2)
18	−0.92401(0)	−0.41392(0)	−0.17498(0)	−0.62279(−1)	−0.10100(−2)
19	−0.89427(0)	−0.40010(0)	−0.16852(0)	−0.59120(−1)	−0.76308(−3)
20	−0.86756(0)	−0.38769(0)	−0.16271(0)	−0.56261(−1)	−0.52937(−3)

†The numbers in parentheses denote powers of ten.

Table 2. Influence coefficients† Λ_n

n	$(L/r_0)/Pe =$ 0.0001	0.001	0.01	0.1	1.0
1	0.037236	0.054007	0.090930	0.17387(0)	0.34472(0)
2	0.027942	0.033945	0.047224	0.74580(−1)	0.84525(−1)
3	0.024956	0.027424	0.032812	0.42311(−1)	0.21067(−1)
4	0.023806	0.024907	0.027239	0.30020(−1)	0.58010(−2)
5	0.023145	0.023457	0.024030	0.23110(−1)	0.16060(−2)
6	0.022696	0.022471	0.021859	0.18567(−1)	0.44474(−3)
7	0.022362	0.021738	0.020259	0.15308(−1)	0.12316(−3)
8	0.022098	0.021158	0.018992	0.12940(−1)	0.34108(−4)
9	0.021881	0.020681	0.017956	0.10978(−1)	0.94457(−5)
10	0.021697	0.020277	0.017081	0.94027(−2)	0.26158(−5)
11	0.021537	0.019926	0.016327	0.81098(−2)	0.72441(−6)
12	0.021396	0.019616	0.015665	0.70300(−2)	0.20061(−6)
13	0.021270	0.019338	0.015075	0.61166(−2)	0.55556(−7)
14	0.021155	0.019086	0.014544	0.53364(−2)	0.15386(−7)
15	0.021050	0.018856	0.014061	0.46651(−2)	0.42608(−8)
16	0.020952	0.018643	0.013618	0.40845(−2)	0.11803(−8)
17	0.020862	0.018445	0.013210	0.35803(−2)	0.32724(−9)
18	0.020778	0.018260	0.012830	0.31409(−2)	0.90767(−10)
19	0.020698	0.018086	0.012477	0.27572(−2)	0.24374(−10)
20	0.020623	0.017922	0.012145	0.24215(−2)	0.80034(−11)

†The numbers in parentheses denote powers of ten.

The influence coefficients can be used to superpose the various pulse arrays of Fig. 2 so as to synthesize the given heat flux distribution. The process follows a pattern identical to that for the case of prescribed temperature, but with one exception. For prescribed periodic wall temperature, the bulk temperature is the same at corresponding axial stations in successive modules. On the other hand, for prescribed periodic heat flux, the bulk temperature will, in general, change from module to module by an amount ΔT_b [equation (2)]. Therefore, it is appropriate to identify the value of the bulk temperature at some reference location.

Since attention is being focused on a module which extends from $x = x^+$ to $x = (x^+ + L)$ as depicted at the top of Fig. 2, the bulk temperature T_b^+ at $x = x^+$ will be used as the reference value. The use of this reference temperature means that appropriate constants have to be added to the influence coefficients. This follows because

$$\frac{T_{wn} - T_b^+}{(qr_0/k)} = \frac{T_{wn} - T_{bn}}{(qr_0/k)} + \frac{T_{bn} - T_b^+}{(qr_0/k)}. \tag{12}$$

The first term on the right is Λ_n, whereas the second term is the change in the bulk temperature between x

$= x^+$ and $x = x_n$. The latter is evaluated from the energy balance

$$\dot{m}c_p \mathrm{d}T_b/\mathrm{d}x = 2\pi r_0 q(x) \quad \text{or} \quad \mathrm{d}T_b/\mathrm{d}X = (4r_0/k)q(X) \quad (13)$$

where $q(X) = q$ at a pulse and is zero otherwise.

Aside from the modification just discussed, the superposition of the various pulse arrays proceeds as before, with the result

$$[T_w(j) - T_b^+]k/r_0 = \sum_{n=1}^{j-1} [\Lambda_{j+1-n} + 4l]/q_n + (\Lambda_1 + 2l)q_j + \sum_{n=j+1}^{N} \Lambda_{N+j+1-n} q_n. \quad (14)$$

The quantity l is the dimensionless pulse width defined by equation (6).

Equation (14) can be employed to evaluate the wall temperature distribution for any prescribed periodic variation of the wall heat flux. A listing of the influence coefficients Λ_n for the circular tube is given in Table 2 for $N = 20$ and for the same values of $(L/r_0)/Pe$ as were used for Table 1.

APPLICATION OF THE SOLUTION METHOD

To illustrate the method, solutions were sought for the periodic wall temperature variation

$$T_w = \sin(2\pi x/L) \quad (15)$$

where the amplitude is chosen as unity for convenience. As noted earlier, a constant can be added to equation (15) without affecting the results. The dimensionless module length $(L/r_0)/Pe$ was selected as 10^{-2}.

The heat flux distribution corresponding to equation (15) was obtained in two ways. One was by using equation (10) in conjunction with Table 1. For the other, the solution was recognized to be of the form

$$T(r, x) = \chi(r)\sin(2\pi x/L) + \psi(r)\cos(2\pi x/L). \quad (16)$$

When equation (16) is substituted into the energy equation (3), a pair of coupled ordinary differential equations are obtained for χ and ψ. These were solved by finite differences, with the result

$$qr_0/k = 6.23\sin(2\pi x/L) + 3.95\cos(2\pi x/L). \quad (17)$$

Figure 4 shows a comparison of the heat flux distribution obtained from equation (10) and Table 1 with that from equation (17). The agreement is seen to be quite satisfactory, especially in view of the minimal computational effort involved in applying equation (10). Furthermore, the influence coefficients Ω_n that were used as input to equation (10) are based on only 20 subdivisions. As will be demonstrated shortly, even better agreement can be obtained when a greater number of subdivisions are employed.

As a second illustration, equation (14) and Table 2 were employed to predict the wall temperature in the presence of a prescribed heat flux. For this purpose, the given heat flux distribution was that of equation (17), which corresponds to the wall temperature distribution (15).

The application of equation (14) and Table 2 to the heat flux (17) yields a wall temperature distribution that is shown in Fig. 5. Also plotted in the figure is the exact solution, equation (15). The agreement in evidence in this figure is even better than that of Fig. 4.

As will be demonstrated in the next section of the paper, influence coefficients for any number of subdivisions N can be generated without difficulty. To examine the effect of the number of subdivisions on the results, the heat flux distribution corresponding to the wall temperature (15) was evaluated from equation (10) using influence coefficients based on $N = 40$ and $N = 80$. These results are plotted in Fig. 6 along with the exact solution (17). It is evident that higher accuracy can be obtained by using a greater number of subdivisions, although $N = 20$ should be sufficient for most applications.

It is interesting to observe that equation (15) and its harmonics, when taken together as a Fourier series, provides an alternative approach to solving problems of prescribed periodic wall temperature. To implement such an approach, it would have been necessary to obtain wall heat flux results similar to equations (17) for all the harmonics. The coefficients in these equations would then be tabulated in a manner similar to the listing of the influence coefficients in Table 1.

The selection of the pulse method in preference to the aforementioned Fourier method was based on the fact that the latter would require an additional preparatory step to initiate its use. Specifically, the user would have to fit a Fourier series to the prescribed wall temperature variation. Furthermore, for many types of variations, especially those involving discrete data or rapid changes, it is not possible to obtain a satisfactory fit with a Fourier series. By comparison, the task of discretizing a given variation into pulse arrays involves virtually no effort. Also, the pulse representation can usually be made to fit the given variation to any degree of accuracy.

DETERMINATION OF THE INFLUENCE COEFFICIENTS

As noted earlier, there are various options available for the determination of the influence coefficients. In certain problems, it may be necessary to solve the energy equation for one typical module in a periodic array of pulses such as those pictured in Fig. 3. On the other hand, in other problems, the influence coefficients may be deduced from available solutions by means of algebraic operations. This latter approach can be employed for determining the influence coefficients for laminar tube flow with prescribed periodic variations of wall temperature or wall heat flux, respectively listed in Tables 1 and 2. The case of prescribed wall temperature will be considered first.

Prescribed wall temperature

In the literature, solutions are available for the fundamental problem of a hydrodynamically developed, isothermal laminar flow with bulk tempera-

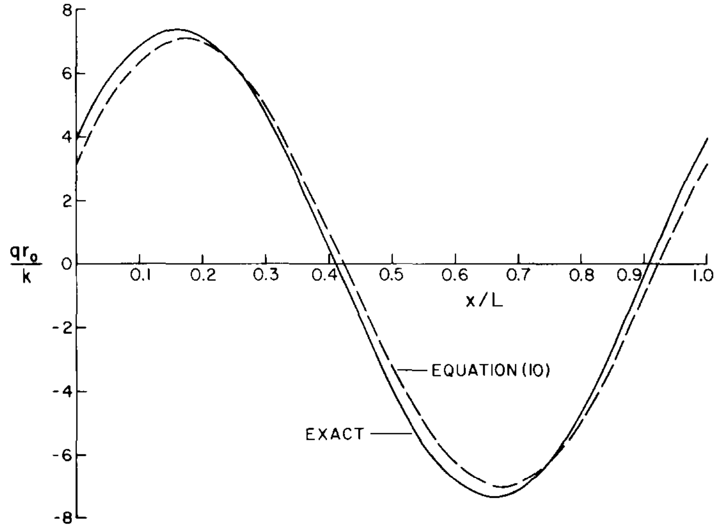

FIG. 4. Wall heat flux distribution corresponding to a sine-wave wall temperature variation. Equation (10) was evaluated with $N = 20$.

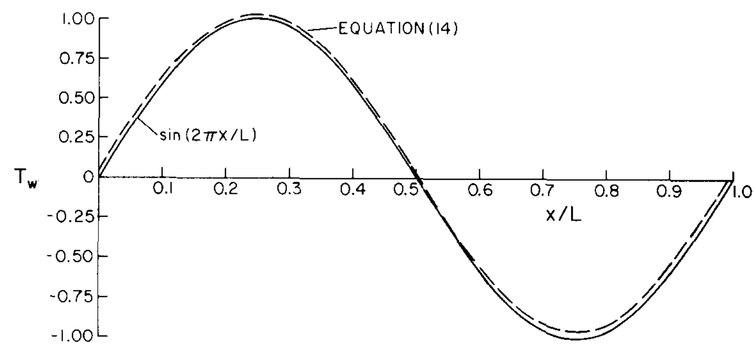

FIG. 5. Wall temperature distribution corresponding to the wall heat flux variation of equation (17). Equation (14) was evaluated with $N = 20$.

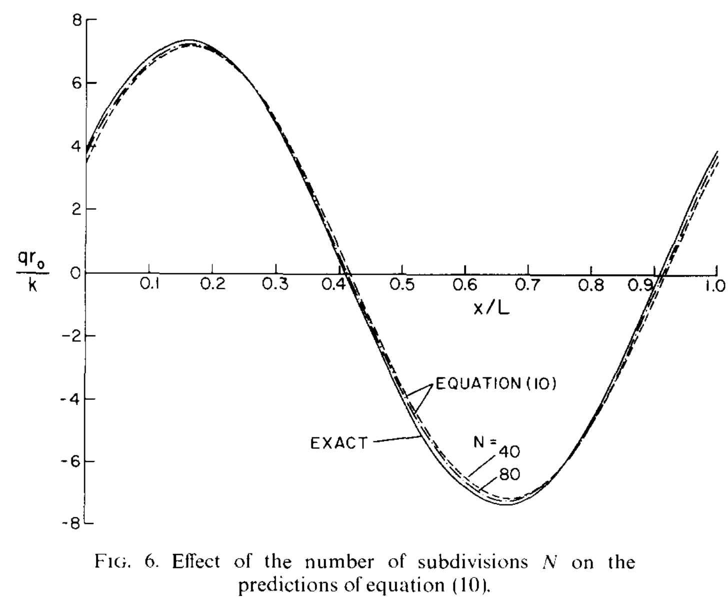

FIG. 6. Effect of the number of subdivisions N on the predictions of equation (10).

ture T_{b0} entering a tube whose wall temperature is uniform and equal to T_w. Since the energy equation which forms the basis of these solutions is linear, it enables superposition of variants of the fundamental problem. Furthermore, without loss of generality, T_{b0} can be taken as the datum temperature (i.e. $T_{b0} = 0$) and T_w referred to it.

For the fundamental problem, the solution for the local wall heat flux has the form

$$\frac{qr_0}{kT_w} = f\left(\frac{\zeta/r_0}{Pe}\right) \tag{18}$$

where ζ is the distance measured downstream from the cross-section where the flow first encounters the wall temperature T_w. The explicit algebraic expressions for f will be presented later. In addition, in view of the argument of f, it is advantageous to work in terms of the dimensionless streamwise coordinate $X = (x/r_0)/Pe$ and, in what follows, all lengths will be dimensionless with respect to $r_0 Pe$.

With regard to superposition, suppose, for example, that the solutions for the following problems are to be added.

(a) Hydrodynamically developed, isothermal flow with bulk temperature $T_{b0} = 0$ for $X < 0$. Tube wall temperature uniform and equal to T_w for $X > 0$.

(b) Hydrodynamically developed, isothermal flow with bulk temperature $T_{b0} = 0$ for $X < l$. Tube wall temperature uniform and equal to $-T_w$ for $X > l$.

The wall temperature distribution for the problem which results from the additive superposition of (a) and (b) is

$$\left.\begin{aligned} T &= 0, \quad X < 0 \\ T &= T_w, \quad 0 < X < l \\ T &= 0, \quad X > l \end{aligned}\right\} \tag{19}$$

which represents a wall temperature pulse of height T_w and duration l.

A pair of such pulses which are spaced by a dimensionless distance $\mathscr{L}$ can be synthesized by adding to (a) and (b) the following:

(c) Same as (a), but with 0 in the X-inequalities replaced by $\mathscr{L}$.

(d) Same as (b), but with l replaced by $(l + \mathscr{L})$.

By proceeding along these lines, a periodic array of pulses of duration l that are separated by a period length $\mathscr{L}$ can be built up. The solution for the array is the sum of the solutions for problems (a), (b), (c), ...

For these pulses, the local heat flux distribution in the periodic thermally developed regime can be deduced by making use of the aforementioned superposition. To facilitate the derivation, reference is made to Fig. 3 which shows the periodically pulsed wall temperature distribution for several modules in the developed regime. Each module is subdivided into N segments of length l such that the period length $\mathscr{L}$ is equal to Nl. The points $1, 2, \ldots, N$ are positioned at the midpoints of the respective segments. Attention is focused on the module delineated by braces.

Consider first the contribution to the heat flux at points $1, 2, 3, \ldots$ due to the pulse at segment 1. This pulse is created by temperature steps of height T_w and $-T_w$, respectively applied at $X = X^*$ and $X = X^* + l$. From equation (18), there follows

$$\left.\begin{aligned} q_1 r_0/kT_w &= f(l/2) \\ q_2 r_0/kT_w &= f(3l/2) - f(l/2) \\ q_3 r_0/kT_w &= f(5l/2) - f(3l/2) \end{aligned}\right\} \tag{20}$$

and so forth.

Next, the contributions to $q_1, q_2, q_3, \ldots$ of the pulse between $X = (X^* - \mathscr{L})$ and $X = (X^* - \mathscr{L}) + l$ can be written

$$\left.\begin{aligned} q_1 r_0/kT_w &= f(\mathscr{L} + l/2) - f(\mathscr{L} - l/2) \\ q_2 r_0/kT_w &= f(\mathscr{L} + 3l/2) - f(\mathscr{L} + l/2) \\ q_3 r_0/kT_w &= f(\mathscr{L} + 5l/2) - f(\mathscr{L} + 3l/2) \end{aligned}\right\} \tag{21}$$

etc. The contributions of all prior pulses follow in a similar manner.

At any point $n = 1, 2, \ldots$, the sum of all such contributions leads to

$$q_n r_0/kT_w = \sum_{m=1} \{f[(m-1)\mathscr{L} + (2n-1)l/2] - f[(m-1)\mathscr{L} + (2n-3)l/2]\} \tag{22}$$

where $f = 0$ when its argument is negative. The upper index of the summation has purposely been left indefinite since the summing operation is continued until q_n is not affected by the use of additional terms.

Equation (22) was employed in the determination of the Ω influence coefficients of Table 1, where $N = \mathscr{L}/l = 20$. It was also employed to generate the influence coefficients for $N = 40$ and 80 used in Fig. 6.

For the evaluation of equation (22), algebraic expressions for f were taken from equation (18) and Table 4 of [3] and from equation (5) and Table 1 of [4]. The first of these is a Leveque-type solution and is especially accurate at small values of the argument of f. It was used for arguments between zero and 0.0076. The second, a Graetz-type solution whose accuracy increases as the argument increases, was used for arguments greater than 0.0076. At the break point, both expressions yield identical values of f.

Prescribed wall heat flux

The influence coefficients for prescribed wall heat flux will be derived by a procedure similar to that of the foregoing. The fundamental problem whose solutions will now be utilized is that in which a hydrodynamically developed, isothermal flow (bulk temperature $T_{b0} = 0$) enters a tube in which there is a uniform wall heat flux q. The solution for the wall temperature distribution has the form

$$\frac{T_w}{(qr_0/k)} = g\left(\frac{\zeta/r_0}{Pe}\right) \tag{23}$$

where ζ is the downstream distance from the cross-section where q is first imposed.

By superposing positive and negative heat flux steps in a manner identical to that of the preceding section, a periodic array of heat pulses of height q, duration l, and period length $\mathscr{L}$ can be constructed. Figure 3 will now be employed as a representation of such an array.

The pulse that is astride the segment 1 is the resultant of heat flux steps of height q and $-q$, respectively applied at $X = X^*$ and $X = X^*+l$. Corresponding to this pulse, the dimensionless wall temperatures $T_{wn}/(qr_0/k)$ at points $1, 2, \ldots$ can be written with the aid of the fundamental solution (23). The resulting algebraic expressions are identical to the RHS of equation (20), with f replaced by g. In the same way, the contribution of the pulse between $X = (X^* - \mathscr{L})$ and $X = (X^* - \mathscr{L})+l$ to $T_{wn}/(qr_0/k)$ is given by the RHS of equation (21), again with f replaced by g. The contributions of all prior pulses can be written in an analogous manner.

The wall temperature distribution in the module of interest can be obtained by summing the various contributions identified in the prior paragraphs with the result

$$T_{wn}/(qr_0/k) = \text{RHS of (22) with } f \to g. \quad (24)$$

It is important to note that T_{wn} will increase steadily as more and more terms are taken in the summation that appears on the RHS of equation (24). This is because the wall temperature itself never becomes fully developed. Rather, it is $(T_w - T_b)$ that is endowed with fully developed characteristics.

It is, therefore, relevant to obtain expressions for $(T_w - T_b)_n/(qr_0/k)$. With regard to the bulk temperature, its variation is governed by

$$\mathrm{d}T_b/\mathrm{d}X = (4r_0/k)q(X) \quad (25)$$

where $q(X) = q$ at the successive pulses and is zero otherwise. From this, it readily follows that

$$T_{bn}/(qr_0/k) = \sum_{m=1} (4l) - 2l\delta_{n1} \quad (26)$$

where $\delta_{n1} = 1$ or 0 when $n = 1$ or $\neq 1$. Then, by bringing together equations (24) and (26),

$$(T_w - T_b)_n/(qr_0/k) = \text{RHS of (24)} - \text{RHS of (26)}. \quad (27)$$

The summing operation in equation (27) is continued until the use of additional terms has no more effect.

For the numerical evaluation of equation (27), the expression for g given in equation (17) and Table 2 of [3] was employed for arguments between zero and 0.033. For arguments greater than this value, the g expression was that of equation (9) and Table 1 of [5].

Equation (27) was used to generate the influence coefficients Λ listed in Table 2. The calculations were performed with $N = \mathscr{L}/l = 20$ and for several parametric values of $\mathscr{L}$.

CONCLUDING REMARKS

It has been demonstrated that a streamwise periodic variation of either the wall temperature or the wall heat flux gives rise to a periodic thermally developed regime at sufficiently large downstream distances. If L denotes the period length of the variation, then in the developed regime successive streamwise modules of length L possess similar heat-transfer characteristics.

A methodology was developed for obtaining universal solutions for the periodic thermally developed regime, either for prescribed wall temperature or prescribed wall heat flux. When the temperature is prescribed, the method enables the distribution of the wall heat flux in any typical period length to be determined via the summing of a simple series. The coefficients in the series (i.e. the influence coefficients) are universal in that they do not depend in any way on the wall temperature distribution. The series will accept any temperature distribution as input. Similarly, a series involving universal coefficients was constructed for determining the wall temperature distribution corresponding to any periodic wall heat flux variation.

The influence coefficients for the case of given temperature stem from the solution of a problem consisting of a periodic array of temperature pulses. Similarly, for given heat flux, the basic problem is that of a periodic array of heat pulses. These problems may be solved by dealing directly with the differential energy equation and using the fact that only a typical module of period length L need be considered for the thermally developed solution.

An alternative procedure for solving the basic pulse problem was developed here which circumvents the task of solving the energy equation. It makes use of available thermal entrance region solutions for uniform wall temperature and uniform wall heat flux. The pulse problem results are obtained by performing summing and differencing operations on the entrance region solutions. This approach was employed to generate the influence coefficients listed in this paper for laminar pipe flow.

The universal solution method was illustrated by application to sine wave temperature and heat flux distributions, and good agreement was obtained with the results of a finite-difference solution.

The methodology developed here is not restricted to laminar tube flows. Rather, it can be employed for any duct flow situation for which the appropriate influence coefficients can be generated.

The use of a Fourier series method as an alternative approach for solving problems of prescribed periodic wall temperature or heat flux was discussed in the text following the illustrative example. The reasons for preferring the pulse method were outlined there.

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LE REGIME PERIODIQUE THERMIQUEMENT DEVELOPPE DANS DES TUBES AVEC TEMPERATURE OU FLUX PARIETAUX PERIODIQUES LONGITUDINALEMENT

Résumé—Un régime périodique thermiquement développé s'établit à partir d'une certaine distance de l'entrée lorsqu'un écoulement dans un tube est soumis à une variation périodique de température de paroi ou de flux pariétal dans le sens du courant. En régime développé, les caractéristiques de transfert se retrouvent semblables dans des modules successifs dont la longueur est égale à la période de variation imposée. On développe une méthodologie pour obtenir des solutions universelles pour le régime périodique pleinement développé. Le résultat final du développement permet d'obtenir la distribution du flux de chaleur, correspondant à une variation quelconque de température pariétale, par sommation d'une simple série. De même est déterminée la température de la paroi quand est donnée la distribution du flux thermique. Les séries contiennent des coefficients d'influence qui représentent la solution pour une suite de pulsations. On présente une procédure pour déterminer ces coefficients d'influence, laquelle évite la résolution de l'équation de l'énergie. L'utilisation de la méthodologie est illustrée par le cas de distributions sinusoidales de la température et du flux thermique à la paroi.

DAS PERIODISCHE THERMISCH AUSGEBILDETE GEBIET IN KANÄLEN MIT IN STRÖMUNGSRICHTUNG PERIODISCHER WANDTEMPERATUR ODER WÄRMESTROMDICHTE

Zusammenfassung—Wenn eine Strömung in einem Rohr oder Kanal in Strömungsrichtung einer periodischen Änderung der Wandtemperatur oder der Wärmestromdichte an der Wand unterworfen wird, so bildet sich in genügendem Abstand vom Einlaß ein periodisches thermisch ausgebildetes Gebiet aus. Im ausgebildeten Gebiet treten in nach Strömungsrichtung aufeinanderfolgenden Abschnitten, deren Länge der Periode der aufgegebenen Änderung entspricht, ähnliche Wärmetransporteigenschaften auf. Es wurde eine Methode entwickelt, um allgemeine Lösungen für das periodische, voll ausgebildete Gebiet zu erhalten. Das Endergebnis der Entwicklung ermöglicht durch Summieren einer einfachen Reihe die Bestimmung der Verteilung der Wärmestromdichte längs der Wand für jede vorgegebene Änderung der Wandtemperatur. Eine ähnliche Reihe ermöglicht die Bestimmung der Wandtemperatur, wenn die Verteilung der Wärmestromdichte vorgegeben ist. Die Reihe enthält Einflußkoeffizienten, welche die Lösung für ein Feld von Impulsen darstellen. Es wird ein Verfahren zur Bestimmung dieser Einflußkoeffizienten vorgestellt, welches die Notwendigkeit der Lösung der differentiellen Energiegleichung umgeht. Die Methode wurde durch Anwendung auf sinusförmige Verteilung der Wandtemperatur und Wärmestromdichte veranschaulicht.

ПЕРИОДИЧЕСКИЙ РАЗВИТЫЙ ТЕПЛОВОЙ РЕЖИМ В КАНАЛАХ С ПЕРИОДИЧЕСКИ ИЗМЕНЯЮЩИМИСЯ ПО НАПРАВЛЕНИЮ ТЕЧЕНИЯ ТЕМПЕРАТУРОЙ СТЕНКИ И ПЛОТНОСТЬЮ ТЕПЛОВОГО ПОТОКА

Аннотация — Когда при течении в трубе температура стенки трубы или плотность теплового потока на стенке периодически изменяются по направлению течения, то периодический развитый тепловой режим устанавливается на достаточно больших расстояниях от входа. При развитом режиме одинаковые коэффициенты теплообмена встречаются в последовательно чередующихся направленных по течению модулях, длина которых равна периоду наложенных возмущений. Разработана методика получения универсальных решений для периодического полностью развитого режима, позволяющая посредством суммирования простых рядов определять распределение плотности теплового потока на стенке при любом заданном изменении температуры стенки. Такие ряды дают возможность определять температуру стенки при заданном распределении плотности теплового потока. В данный ряд входят коэффициенты влияния, представляющие решение для любого вида импульсов. Предложена методика определения этих коэффициентов, при которой отпадает необходимость решения дифференциального уравнения энергии. Применение методики показано на примере синусоидальных изменений температуры стенки и плотности теплового потока.